*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. a) Let $f = f(x, y, z)$, $X = X(u, v, w)$, $Y = Y(u, v, w)$, $Z = Z(u, v, w)$ be functions of three variables. Consider the composite function $F(u, v, w)$,

$$F(u, v, w) = f\left(X(u, v, w),\, Y(u, v, w),\, Z(u, v, w)\right).$$

Write down the formulae for the partial derivatives $\dfrac{\partial F}{\partial u}, \dfrac{\partial F}{\partial v}, \dfrac{\partial F}{\partial w}$ in terms of $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ and $\dfrac{\partial X}{\partial u}, \dfrac{\partial X}{\partial v}, \dfrac{\partial X}{\partial w}, \dfrac{\partial Y}{\partial u}, \dfrac{\partial Y}{\partial v}, \dfrac{\partial Y}{\partial w}, \dfrac{\partial Z}{\partial u}, \dfrac{\partial Z}{\partial v}, \dfrac{\partial Z}{\partial w}$.

b) Let $f(x, y, z) = xyz$, $X(u, v, w) = u + v$, $Y(u, v, w) = u - v$, $Z(u, v, w) = w^2$. Consider $F(u, v, w) = f(X(u, v, w),\, Y(u, v, w),\, Z(u, v, w))$.
Find $\dfrac{\partial F}{\partial u}(u, v, w)$, $\dfrac{\partial F}{\partial v}(u, v, w)$, $\dfrac{\partial F}{\partial w}(u, v, w)$ using the chain rule.

c) Let $R$ be a region on the $xy$ plane defined by

$$x \leqslant y \leqslant x + 2, \quad -x \leqslant y \leqslant 2 - x\ .$$

Find the integral

$$\iint_R (x - y)\,dxdy.$$

2. a) For the surface given as the graph of the function $f(x, y) = (x + y)e^{x+y}$ find the equation of its tangent plane at the point $(1, -1, 0)$.

b) Let $\mathbf{u} = (u_1, u_2)$ be a unit vector, $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1$. Let $f(x, y)$ be a function of two variables.
Define the directional derivative $\dfrac{\partial f}{\partial \mathbf{u}}$.

c) Let a surface $S$ be given as the graph of the function $f(x, y) = 5 - x^2 - y^2$, where $(x, y)$ satisfy

$$x^2 + y^2 \leqslant 1, \quad y \geqslant x,\ x \geqslant 0.$$

Find the surface integral

$$\iint_S \frac{x^2 - y^2}{x^2 + y^2}\, dS.$$

3. a) State the Divergence Theorem carefully.

   b) Verify the Divergence Theorem when the vector-field $\mathbf{F}$ has the form,

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + (x^2 + y^2)\mathbf{k},$$

   and $D$ is defined as

$$D = \{(x, y, z) : \quad x^2 + y^2 \leqslant 1, \quad 0 \leqslant z \leqslant \sqrt{1 - x^2 - y^2}\ \} .$$

4. a) State Green's Theorem in the plane carefully.

   b) Verify Green's Theorem in the plane for the vector-field

$$\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$$

   and the region $R$ defined by

$$x^2 + y^2 \leqslant 1, \quad y \leqslant 0.$$

   c) Let the contour $C$ be the boundary of the lower half-disk of radius 1 centered at the origin, and oriented in the anti-clockwise direction.
   Let

$$\mathbf{F}(x, y) = (y^2 e^x \cos(e^x) - y)\mathbf{i} + 2(y \sin(e^x) + x)\mathbf{j}.$$

   Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$.
   Hint: You can use Green's Theorem.

5. a) Let $D = \mathbb{R}^3 \setminus \{(x, y, z) : x = y = z = 0\}$. Let $C$ be a smooth curve in $D$. Let $\mathbf{F}(x, y, z)$ be a smooth vector field in $D$ such that

$$\nabla \times \mathbf{F} = \mathbf{0}, \quad \text{in } D.$$

Does it guarantee that $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the initial point $\mathbf{X}_0$, and terminal point $\mathbf{X}_1$ of the curve $C$? Explain your answer.

b) Let

$$\mathbf{F} = \left(y^2 \cos(xy^2) + 2xy^2(1 + x^2y^2)^{-1}\right) \mathbf{i} + \left(2xy \cos(xy^2) + 2x^2y(1 + x^2y^2)^{-1}\right) \mathbf{j} + z\mathbf{k}.$$

Show that $\nabla \times \mathbf{F} = \mathbf{0}$ and find a potential function for $\mathbf{F}$.

c) Let

$$\mathbf{G} = \left(y^2 \cos(xy^2) + 2xy^2(1 + x^2y^2)^{-1} + y\right) \mathbf{i} + \left(2xy \cos(xy^2) + 2x^2y(1 + x^2y^2)^{-1} + y\right) \mathbf{j}.$$

Let $C$ be a curve of the form $x = t$, $y = 1 - t$, $z = 2 - t$, with $0 \leqslant t \leqslant 1$, which connects the point $\mathbf{X}_0 = (0, 1, 2)$ with the point $\mathbf{X}_1 = (1, 0, 1)$.
Find

$$\int_C \mathbf{G} \cdot d\mathbf{r}.$$

[Hint: You can use Part b).]

6. a) State Fourier's Theorem carefully.

b) Find the Fourier coefficients of the function $f(x)$ which

- is equal to $|x|$ for $-\pi/2 \leqslant x < \pi/2$;
- continued $\pi$ periodically from $[-\pi/2, \pi/2)$ to the whole $\mathbb{R}$.